

**An Introduction to PALM:  
Filtering – Filter methods  
Subgridscale models**

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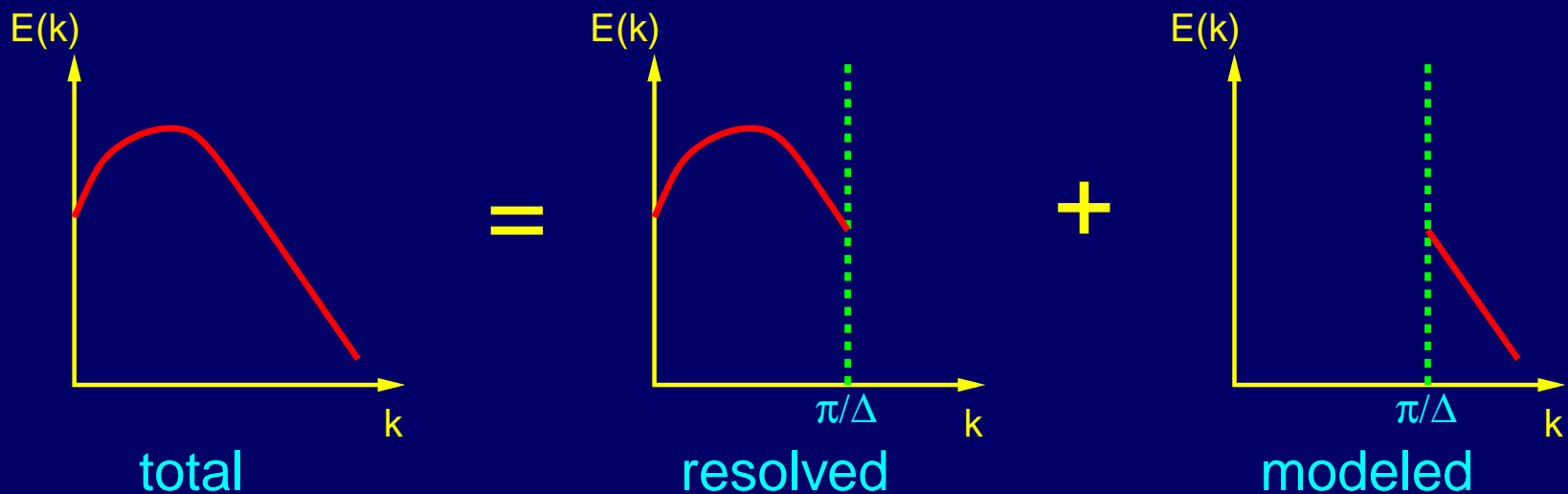
# Overview/Content

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# Introduction

- ⇒ The large-eddy simulation technique is based on scale separation, in order to reduce number of degrees of freedom of the solution
- ⇒ Low-frequency modes are calculated directly (resolved scales)
- ⇒ Small/subgrid/subfilter scales are parameterized using a statistical model (sub-grid/sub-filter model)
- ⇒ These two categories of scales are separated by defining a cutoff length  $\Delta$



# Filtered Navier-Stokes equations

⇒ Applying the filter to the Navier-Stokes equations leads to

**Momentum equations:**

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\overline{u_j u_i}) = -\frac{1}{\rho_0} \frac{\partial \bar{p}^*}{\partial x_i} + \frac{g}{\theta_{v00}} (\bar{\theta}_v - \theta_{v00}) \delta_{i3}.$$

**Equation of continuity:**

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

**Equations for scalar variables:**

$$\frac{\partial \bar{\psi}}{\partial t} = -\frac{\partial (\overline{u_i \psi})}{\partial x_i}$$

⇒ The filtering process yields non-linear terms, e.g.  $\overline{u_j u_i}$ ,

⇒ In order for these equations to be usable the non-linear terms (e.g.,  $\overline{u_j u_i}$ ) have to be expressed as a function of the filtered quantities (e.g.,  $\bar{u}_i$ ) and of sub-filter scales ( $u'_i$ ) (decomposing the non-linear terms)

# Decomposing non-linear terms – Leonard's decomposition (I)

⇒ Introducing the basic definition of filtered and sub-filter scales the non-linear term could be expressed as a triple summation:

$$\overline{u_i u_j} = \overline{(\bar{u}_i + u'_i)(\bar{u}_j + u'_j)} = \overline{\bar{u}_i \bar{u}_j} + \underbrace{\overline{\bar{u}_i u'_j} + \overline{u'_i \bar{u}_j}}_{C_{ij}} + \underbrace{\overline{u'_i u'_j}}_{R_{ij}}$$

⇒ Non-linear term is entirely written as a function of filtered and sub-filter scales ( $\bar{u}_i$  and  $u'_i$ )

⇒ Two versions of introducing this decomposition to the Navier-Stokes equations exist:

- ① Double decomposition (straight forward solution, not considered here)
- ② Triple decomposition

# Decomposing non-linear terms – Leonard's decomposition (II)

## Triple decomposition

☞ Considers that it must be possible to evaluate the terms directly from the filtered variables

☞  $\overline{u_i u_j}$  cannot be calculated directly, requires a second application of the filter

☞ Leonard proposes a further decomposition

$$\overline{u_i u_j} = (\overline{\overline{u_i u_j}} - \overline{u_i u_j}) + \overline{u_i u_j} = L_{ij} + \overline{u_i u_j}.$$

☞ Defining  $\tau_{ij}$  as

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij} = \overline{\overline{u_i u_j}} - \overline{u_i u_j}$$

leads to

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial}{\partial x_j} (\overline{u_j u_i}) = -\frac{1}{\rho_0} \frac{\partial \overline{p}^*}{\partial x_i} + \frac{g}{\theta_{v00}} (\overline{\theta_v} - \theta_{v00}) \delta_{i3} - \frac{\partial \tau_{ij}}{\partial x_j}$$

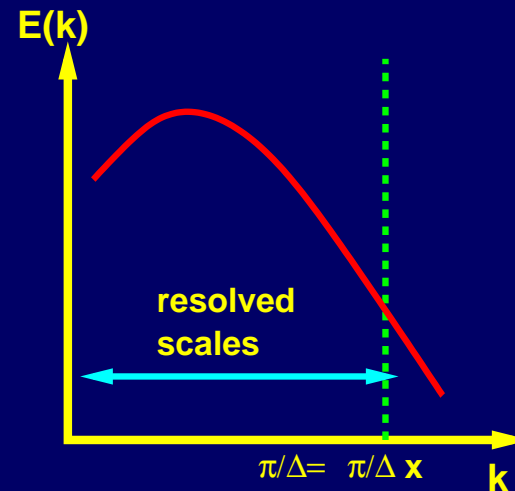
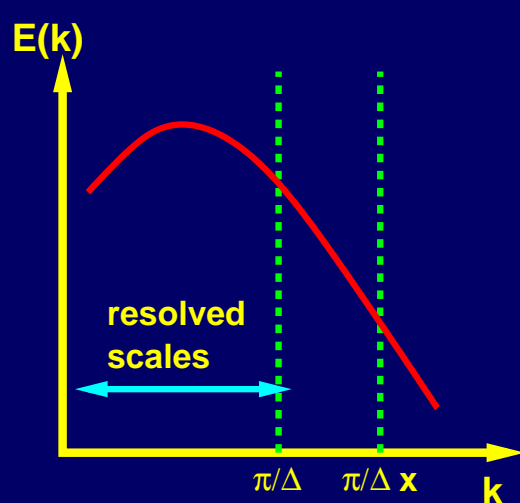
# Intermediate summary

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- ⇒ The large-eddy simulation technique is a technique for reducing the number of degrees of freedom by separating the scales
- ⇒ The selection is made by a filtering technique (low-pass filter)
- ⇒ Two categories: resolved scales and sub-grid scales
- ⇒ The complexity of the solution is reduced by retaining only the large scales
- ⇒ The information concerning the small scales is contained in the sub-grid tensor
- ⇒ In order for the dynamics of the resolved scales to remain correct, the sub-grid terms have to be considered ⇒ sub-grid models

# Theory and practice

- ⇒ The previous developments completely ignore the computational grid used for solving the Navier-Stokes equations numerically
- ⇒ The computational grid introduces another space scale: the discretization step  $\Delta x$
- ⇒  $\Delta x$  has to be small enough to be able to apply the filtering process correctly:  
 $\Delta x \leq \Delta$
- ⇒ Two possibilities:
  - ① Pre-filtering technique ( $\Delta x < \Delta$ )
  - ② Linking the analytical filter to the computational grid ( $\Delta x = \Delta$ )



# Theory and practice (II)

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## Pre-filtering:

- ⇒ Increases the number of degrees in the numerical solution without increasing the number of degree of freedom in the physically resolved solution
- ⇒ Requires that the analytical filter is performed explicitly
- ⇒ Rarely used in practice, due to additional computational costs

## Linking the analytical filter to the computational grid:

- ⇒ The analytical cutoff length is associated with the grid spacing
- ⇒ This method does not require the use of an analytical filter
- ⇒ Brings some disadvantages concerning the evaluation of sub-grid models (should not be discussed further)
- ⇒ Because of its simplicity, this method is used by nearly all authors (e.g., see Schumann, 1975)

# The final set of equations:

## Momentum equations:

$$\frac{\partial \bar{u}_i}{\partial t} = -\frac{\partial(\bar{u}_j \bar{u}_i)}{\partial x_j} - \frac{1}{\rho_0} \frac{\partial \bar{p}^*}{\partial x_i} - (\varepsilon_{ijk} f_j \bar{u}_k - \varepsilon_{i3k} f_3 u_{kgeo}) + \frac{g}{\theta_{v00}} (\bar{\theta}_v - \theta_{v00}) \delta_{i3} - \frac{\partial \tau_{ij}}{\partial x_j}$$

## Equation of continuity:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (\text{flow is divergence free by assumption, assured by pressure solver})$$

## Liquid water potential temperature:

$$\frac{\partial \bar{\theta}_l}{\partial t} = -\frac{\partial(\bar{u}_i \bar{\theta}_l)}{\partial x_i} - \frac{\partial W_i}{\partial x_i} + \left( \frac{\partial \bar{\theta}_l}{\partial t} \right)_{\text{rad}} + \left( \frac{\partial \bar{\theta}_l}{\partial t} \right)_{\text{prec}}$$

## Total water content:

$$\frac{\partial \bar{q}}{\partial t} = -\frac{\partial(\bar{u}_i \bar{q})}{\partial x_i} - \frac{\partial H_i}{\partial x_i} + \left( \frac{\partial \bar{q}}{\partial t} \right)_{\text{prec}}$$

$$W_i = \overline{u_i \theta_l} - \bar{u}_i \bar{\theta}_l$$

$$H_i = \overline{u_i q} - \bar{u}_i \bar{q}$$

$$\tau_{ij} = \overline{u_j u_i} - \bar{u}_j \bar{u}_i$$

parameterized

sub-grid model

# Literature

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**Sagaut, P, 2001:** Large eddy simulation for incompressible flows: An introduction.  
— Springer Verlag, Berlin/Heidelberg/New York, 319 p.

**Schumann, U., 1975:** Subgrid scale model for finite difference simulations of  
turbulent flows in plane channels and annuli. – J. Comp. Phys., **18**, 376–404