An Introduction to PALM:
Filtering – Filter methods

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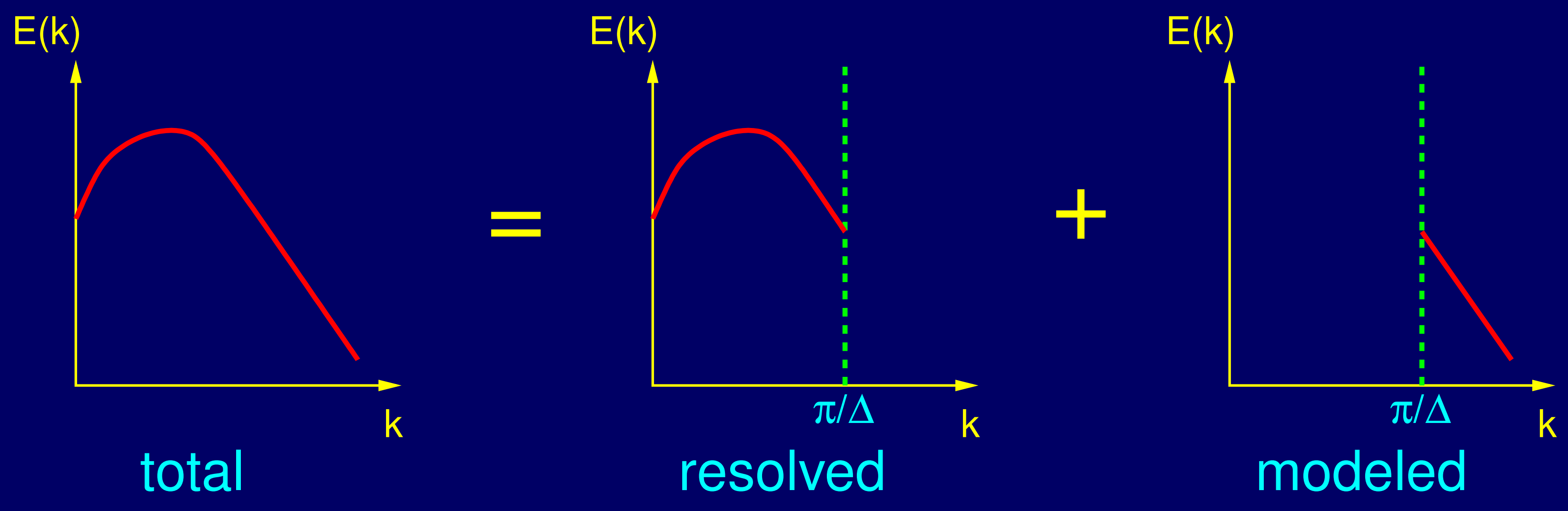
Overview/Content

Introduction

- The large-eddy simulation technique is based on scale separation, in order to reduce number of degrees of freedom of the solution
- Low-frequency modes are calculated directly (resolved scales)
- Small/subgrid/subfilter scales are parameterized by a way of a statistical model (sub-grid/sub-filter model)
- These two categories of scales are separated by defining a cutoff length $\overline{\Delta}$



Theory of Scale Separation – Definitions

- Formally the scales are separated by applying a low-pass filter to the exact solution
- Filtering is mathematically represented by a convolution product in the physical space
- the resolved part is formally defined by the relation:

$$\overline{\phi}(x_i, t) = \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} d^3x_i' \, \phi(x_i', t) \, G(x_i - x_i', t - t') \,, \quad (\overline{\phi} = G \star \phi)$$

with: $G =$ convolution kernel

$\phi =$ exact solution for a space-time variable

- The unresolved (modeled) part is defined as:

$$\phi'(x_i, t) = \phi(x_i, t) - \overline{\phi}(x_i, t) \,, \quad (\phi' = (1 - G) \star \phi)$$

Theory of Scale Separation – Properties of the filter

- The filter has to verify the following properties

  ① Conservation of constants: $$\overline{a} = a \Leftrightarrow \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} d^3x'_i \, G(x'_i, t') = 1 \, .$$

  ② Linearity: $$\overline{\phi + \psi} = \overline{\phi} + \overline{\psi} \, .$$

  ③ Commutation with derivation: $$\overline{\frac{\partial \phi}{\partial s}} = \frac{\partial \overline{\phi}}{\partial s} \, , \qquad s = x_i, \, t \, .$$

- Filters verifying these properties are not, in the general case, Reynolds operators ($\overline{\overline{\phi}} = \overline{\phi}$, $\overline{\phi'} = 0$):

$$\begin{aligned} \overline{\overline{\phi}} &= G \star G \star \phi = G^2 \star \phi \neq \overline{\phi} \\ \overline{\phi'} &= G \star (1 - G) \star \phi \neq 0 \end{aligned}$$

Classical filters for Large-Eddy Simulation

- Box or "“top-hat"” filter:

$$G(x-x') = \begin{cases} \frac{1}{\overline{\Delta}} & \text{if } \left|x-x'\right| \le \frac{\overline{\Delta}}{2} \\ 0 & \text{else} \end{cases}, \qquad \widetilde{G}(k) = \frac{\sin(k\overline{\Delta}/2)}{(k\overline{\Delta})/2} .$$

- Gaussian filter:

$$G(x-x') = \left(\frac{\gamma}{\pi\overline{\Delta}^2}\right)^{1/2} \exp\left(\frac{-\gamma\left|x-x'\right|^2}{\overline{\Delta}^2}\right), \qquad \widetilde{G}(k) = \exp\left(\frac{-\overline{\Delta}^2 k^2}{4\gamma}\right),$$

(The constant γ is generally taken to be 6)

- Spectral or sharp cutoff filter:

$$G(x-x') = \frac{\sin(k_c(x-x'))}{(k_c(x-x'))} \quad \text{with } k_c = \frac{\pi}{\overline{\Delta}}, \qquad \widetilde{G}(k) = \begin{cases} 1 & \text{if } |k| \le k_c \\ 0 & \text{else} \end{cases} .$$

Filtered Navier-Stokes equations

- Applying the filter to the Navier-Stokes equations lead to
  Momentum equations:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_j u_i}) = -\frac{1}{\varrho_0}\frac{\partial \overline{p}}{\partial x_i} + \frac{g}{\theta_{v00}}(\overline{\theta}_v - \theta_{v00})\delta_{i3} \, .$$

  Equation of continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

  Equations for scalar variables:

$$\frac{\partial \overline{\psi}}{\partial t} = -\frac{\partial (\overline{u_i \psi})}{\partial x_i}$$

- The filtering process provides the non-linear terms, e.g. $\overline{u_j u_i}$,
- In order for this equations to be usable the non-linear terms (e.g., $\overline{u_j u_i}$) have to be expressed as a function of the filtered quantities (e.g., $\overline{u}_i$) and of sub-filter scales (u_i')
  - This requirement leads to the approach of decomposing the non-linear terms

Decomposing the non-linear term – Leonard's decomposition (I)

- Introducing the basic definition of filtered and sub-filter scales the non-linear term could be expressed as a triple summation:

$$\overline{u_i u_j} = \overline{(\overline{u}_i + u'_i)(\overline{u}_j + u'_j)} = \overline{\overline{u}_i \overline{u}_j} + \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i} + \overline{u'_i u'_j}$$

- Non-linear term is entirely written as a function of filtered and sub-filter scales ($\overline{u}_i$ and u'_i)
- Two versions of introducing this decomposition to the Navier-Stokes equations exist:
  ① Double decomposition (straight forward solution)
  ② Triple decomposition

Decomposing the non-linear term – Leonard's decomposition (II)

① Double decomposition

☞ This approach considers that all the terms appearing in the evolution equation of the filtered quantity must themselves be filtered quantities:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{\overline{u}_j \overline{u}_i}) = -\frac{1}{\varrho_0}\frac{\partial \overline{p}}{\partial x_i} + \frac{g}{\theta_{v00}}(\overline{\theta}_v - \theta_{v00})\delta_{i3} - \frac{\partial \tau_{ij}}{\partial x_j}$$

with τ_{ij} being the sub-grid tensor

☞ τ_{ij} is grouping together terms that are not exclusively dependent on large scales:

$$\tau_{ij} = C_{ij} + R_{ij} = \overline{u_i u_j} - \overline{\overline{u}_i \overline{u}_j}$$

with $C_{ij} = \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i}$ = cross-stress tensor

☛ Reflecting interactions of large and small scales

$R_{ij} = \overline{u'_i u'_j}$ = Reynolds sub-grid tensor

☛ Reflecting interactions between sub-grid scales

Decomposing the non-linear term – Leonard's decomposition (III)

② Triple decomposition

- Considers that it must be possible to evaluate the terms directly from the filtered variables
  - $\overline{\overline{u}_i\overline{u}_j}$ cannot be calculated directly, requires a second application of the filter
- Leonard proposes a further decomposition

$$\overline{\overline{u}_i\overline{u}_j} = (\overline{\overline{u}_i\overline{u}_j} - \overline{u}_i\overline{u}_j) + \overline{u}_i\overline{u}_j = L_{ij} + \overline{u}_i\overline{u}_j\ .$$

- Defining τ_{ij} as

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij} = \overline{u_i u_j} - \overline{u}_i\overline{u}_j$$

leads to

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u}_j\overline{u}_i) = -\frac{1}{\varrho_0}\frac{\partial \overline{p}}{\partial x_i} + \frac{g}{\theta_{v00}}(\overline{\theta}_v - \theta_{v00})\delta_{i3} - \frac{\partial \tau_{ij}}{\partial x_j}$$

Decomposing the non-linear term

- Beside the Leonard decomposition a more general approach exist: Germano consistent decomposition (not shown here)
- For further information it is referred to Literature

Intermediary summary

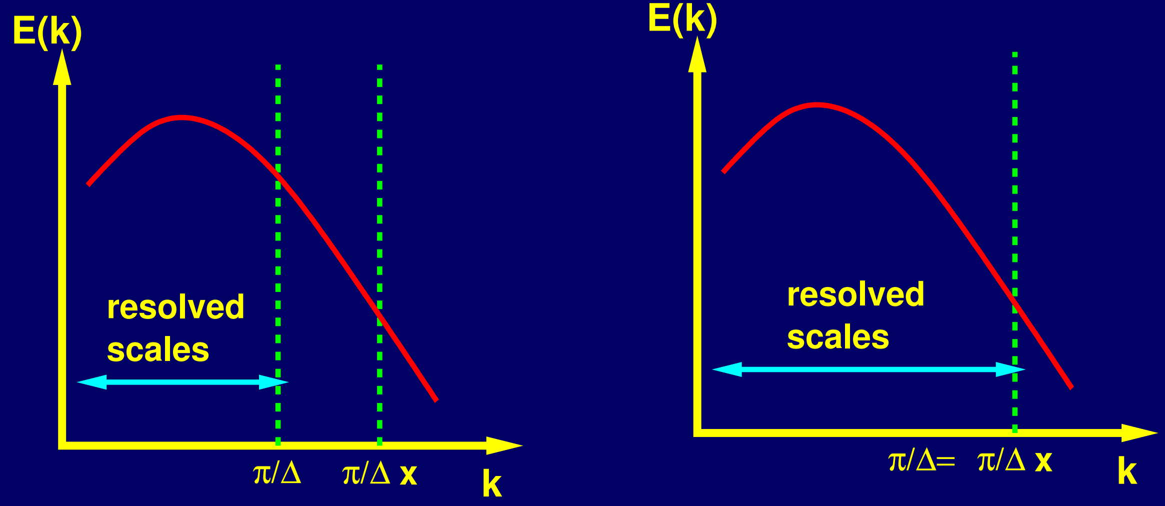
- The large-eddy simulation technique is a technique for reducing the number of degrees of freedom by separating the scales
- The selection is made by a filtering technique (low-pass filter)
- Two categories: resolved scales and sub-grid scales
- The complexity of the solution is reduced by retaining only the large scales
- The information concerning the small scales are grouped into the sub-grid tensor
- In order for the dynamics of the resolved scales to remain correct, the sub-grid terms have to be considered ➪ sub-grid models (see lecture sub-grid-scale models)

Theory and practice

- The previous developments completely ignore the computational grid used for solving the Navier-Stokes equations numerically
- The computational grid introduces another space scale: the discretization step Δx
- Δx has to be small enough to be able to apply the filtering process correctly:

$$\Delta x \leq \overline{\Delta}$$

- Two possibilities:
  ① Pre-filtering technique ($\Delta x < \overline{\Delta}$)
  ② Linking the analytical filter to the computational grid ($\Delta x = \overline{\Delta}$)



Theory and practice (II)

Pre-filtering:

- Increases the number of degrees in the numerical solution without increasing the number of degree of freedom in the physically resolved solution
- Requires that the analytical filter is performed explicitly
- Rarely used in practice, due to additional computational costs

Linking the analytical filter to the computational grid:

- The analytical cutoff length is associated with the grid spacing
- This method does not require the use of an analytical filter
- Brings some disadvantages concerning the evaluation of sub-grid models (should not be discussed further)
- Because of its simplicity, this method is used by nearly all authors (e.g., see Schumann, 1975)

Interpretation of the LES

Understanding:

The energy spectrum of the computed solution has to verify the Kolmogorov spectrum modified by the transfer function of the filter

Kolmogorov spectrum: $E(k) = F(k^{-5/3})$

modified spectrum: $E(k) = F(k^{-5/3} \cdot \widetilde{G}^2)$

Observation:

Applying a ‘backward’ filtering to the computed solution does not reproduce the exact solution

The comparison of spectra calculated from LES-data and theoretical spectra has shown significant differences

Reason:

Actually, the computed solution is the result of four filtering processes in total:

① The analytical filter, used to express the filtered Navier-Stokes equations

② A filter associated with the computational grid (frequencies larger than the Nyquist frequency cannot be represented in the simulation)

③ A filter induced by the numerical scheme (e.g., numerical diffusion of advection schemes)

④ A filter associated with the sub-grid model

Conclusion:

Details of the filtering process, especially the character of the filter-operator, are not known a priori, since the solution obtained by LES is a result of an additional implicit filtering operation

☞ Actually, you don’t know which filter function you are using!

The final set of equations:

Equation of continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Momentum equations:

$$\frac{\partial \overline{u}_i}{\partial t} = -\frac{\partial(\overline{u}_j \overline{u}_i)}{\partial x_j} - \frac{1}{\varrho_0}\frac{\partial \overline{p}^*}{x_i} - (\varepsilon_{ijk} f_j \overline{u}_k - \varepsilon_{i3k} f_3 u_{k_{geo}}) + \frac{g}{\theta_{v00}}(\overline{\theta}_v - \theta_{v00})\delta_{i3} - \frac{\partial \tau_{ij}}{\partial x_j}$$

Liquid water potential temperature:

$$\frac{\partial \overline{\theta}_l}{\partial t} = -\frac{\partial(\overline{u}_i \overline{\theta}_l)}{\partial x_i} - \frac{\partial W_i}{\partial x_i} + \left(\frac{\partial \overline{\theta}_l}{\partial t}\right)_{\mathrm{rad}} + \left(\frac{\partial \overline{\theta}_l}{\partial t}\right)_{\mathrm{prec}}$$

Total water content:

$$\frac{\partial \overline{q}}{\partial t} = -\frac{\partial(\overline{u}_i \overline{q})}{\partial x_i} - \frac{\partial H_i}{\partial x_i} + \left(\frac{\partial \overline{q}}{\partial t}\right)_{\mathrm{prec}}$$

$$\begin{aligned} \tau_{ij} &= \overline{u_j u_i} - \overline{u}_j \overline{u}_i \\ H_i &= \overline{u_i \theta_l} - \overline{u}_i \overline{\theta}_l \\ W_i &= \overline{u_i q} - \overline{u}_i \overline{q} \end{aligned}$$

explicitly solved

parameterized (sub-grid model)

implicitly achieved

Literature

Sagaut, P, 2001: Large eddy simulation for incompressible flows: An introduction. — Springer Verlag, Berlin/Heidelberg/New York, 319 p.

Schumann, U., 1975: Subgrid scale model for finite difference simulations of turbulent flows in plane channels and annuli. – J. Comp. Phys., **18**, 376–404